Substitution $\rho = {}^1/_4 r^2$ leads to the Emden-Fowler equation with a different value of σ

$$\frac{d}{dr}\left(r\frac{dw}{dr}\right) - \gamma r w^{-1} = 0 \qquad (2.8)$$

The boundary conditions now are: $w(0)$ bounded and $w(\lambda l) = 1$ if (2.5) is used, and $w(2\sqrt{\lambda l}) = 1$ in the case (2.6). Dependence of the solutions on the parameter corresponds to Fig. 6a ($\gamma = 1$, attraction), or to Fig. 6c ($\gamma = -1$, repulsion). In the latter case $w_m = w(0)$ is the maximum value of w. These curves can be used in assessing the stability of the equilibrium [5]. In addition, in the repulsion case nonplanar forms of equilibrium and formal solutions describing forms with an apex analogous to those obtained in [5], are also possible. If the dividing curve corresponds to the "usual" forms, then the forms with an apex will have a corresponding combination of the dividing curve with a curve on which $\vartheta \to -\infty$ as $\eta \to 0$ (Fig. 3).

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ON THE MAXIMUM VALUE OF THE RADIUS OF THE CONTACT AREA BETWEEN A PUNCH AND A LAYER

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V. D. LAMZIUK and A. K. PRIVARNIKOV
(Dnepropetrovsk)
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The solution of the following problem of the elasticity theory is given for an infinite weightless homogeneous isotropic layer: a normal concentrated force acts at one of the boundaries of the layer, pressing it against a rigid smooth punch, represented by a convex body of revolution whose axis coincides with the support line of the concentrated force; one has to determine the largest possible value of the radius of the contact area between the punch and the layer for different punches and for different magnitudes of the concentrated force.

1. We consider the layer in a cylindrical system of coordinates $r\theta z$, with origin at the point of application of the concentrated normal force Q. The z-axis is pointed perpendicular to the boundary of the layer in the direction opposite to the direction of the force. Then, the equation of the boundary of the layer, which is in contact with the punch, is $z = -h$, where h denotes the thickness of the layer. We will assume that the base of the punch is a convex surface given by the equation $z = f(r)$, while the lateral surface is cylindrical with radius R. We consider that the function $f(r)$ has in the interval $[0, R]$ at least a second continuous derivative. As a consequence of the convexity of the punch, the area of contact with the layer is a circle whose radius a satisfies the inequality $a \leqslant R$. The case $a = R$ will be called the case of complete indentation of the punch into the layer.

For the normal component of the displacement of the points of the boundary $z = -h$ of the layer and for the normal and shearing stresses on this boundary, we introduce the notation $w(r)$, $\sigma_z(r)$, $\tau_{rz}(r)$. Then, according to the conditions of the problem, for $z = -h$

$$dw / dr = f'(r) \qquad (0 \leqslant r < a) \tag{1.1}$$

$$\sigma_z(r) = 0 \qquad (r > a) \tag{1.2}$$

$$\tau_{rz}(r) = 0 \qquad (0 \leqslant r < \infty) \tag{1.3}$$

Based on the method for solving boundary value problems for a layer in the case of an axially-symmetric deformation [1, 2], it is easy to obtain for the case under consideration the following integral representations for the functions dw / dr and $\sigma_z(r)$ for $z = -h$:

$$\frac{dw}{dr} = \frac{(1-\nu^2)\,2}{E} \int_0^\infty p\, \frac{Q\pi^{-1}(\operatorname{sh} ph + ph \operatorname{ch} ph) - \alpha(p)(\operatorname{sh} 2ph + 2ph)}{\operatorname{ch} 2ph - 2p^2h^2 - 1} J_1(pr)\,dp$$

$$\sigma_z(r) = -\int_0^\infty p\alpha(p) J_0(pr)\,dp, \qquad \alpha(p) = -\int_0^\infty r\sigma_z(r) J_0(pr)\,dr \tag{1.4}$$

In the derivation of the formulas (1.4), one assumes that the condition (1.3) is satisfied at the boundaries of the layer. The improper integral in the first relation of (1.4) converges only if $\alpha(0) = {}^1/_2\, Q\pi^{-1}$ which, however, follows from the condition of equilibrium of the layer ($\Sigma F_z = 0$) and therefore it is always satisfied.

In the boundary conditions (1.1) and (1.2) we substitute for dw / dr and $\sigma_z(r)$ the corresponding integrals in (1.4). We arrive then to dual integral equations with respect to the function $\alpha(p)$, which after elementary transformations can be reduced to the form

$$\int_0^\infty p\alpha(p) J_1(pr)\,dp = -\frac{E}{2(1-\nu^2)} f'(r) +$$

$$+ \int_0^\infty p\, \frac{\alpha(p)(e^{-2ph} - 2p^2h^2 - 2ph - 1) + Q\pi^{-1}(\operatorname{tsh} ph + ph \operatorname{ch} ph)}{\operatorname{ch} 2ph - 2p^2h^2 - 1} J_1(pr)\,dp \qquad (r < a)$$

$$\int_0^\infty p\alpha(p) J_0(pr)\,dp = 0 \qquad (r > a) \tag{1.5}$$

We seek the solution of the dual equations (1.5) in the form

$$p\alpha(p) = \int_0^a \chi(x) \sin px\,dx + B \sin pa \tag{1.6}$$

where B is for the present an arbitrary constant. For the validity of the following computations it is necessary to assume that $\chi(x)$ is a continuous function in the interval $[0,a]$.

By the usual method, based on properties of Bessel functions, the dual equations (1.5) can be reduced to Abel's integral equation

$$\int_0^r \frac{F(x)\,dx}{r\sqrt{r^2-x^2}} = -\frac{E}{2(1-\nu^2)} f'(r)$$

$$F(x) = x\left[\chi(x) - \frac{2}{\pi}\int_0^\infty p\,\frac{\alpha\,(e^{-2ph}-2p^2h^2-2ph-1)+\frac{Q}{\pi}(\operatorname{sh} ph + ph\operatorname{ch} ph)}{\operatorname{ch} 2ph - 2p^2h^2 - 1}\times\right.$$
$$\times \sin px dp$$

The solution of this equation is [3]

$$F(x) = -\frac{E}{2(1-\nu^2)}\frac{2}{\pi}\frac{d}{dx}\int_0^x \frac{r^2 f'(r)\,dr}{\sqrt{x^2-r^2}}$$

In the right-hand side of the last equality we integrate by parts, we differentiate the obtained expression with respect to x and we perform the change of variables $x = r\sin\gamma$. In the expression for $F(x)$ we make use of the relation (1.6) and then we interchange the order of integration in the improper integral. As a result, we arrive at the integral equation of the second kind

$$\chi(x) = -\frac{E}{\pi(1-\nu^2)}\int_0^{{}^1\!/_2\pi} \frac{d}{dr}[rf'(r)]_{r=x\sin\gamma}\,d\gamma + BK(x,a) + \int_0^a \chi(t)\,K(x,t)\,dt \tag{1.7}$$

$$K(x,t) = \frac{2}{\pi}\int_0^\infty \frac{(e^{-2ph}-2p^2h^2-2ph-1)\sin pt + 2pt(\operatorname{sh} ph + ph\operatorname{ch} ph)}{\operatorname{ch} 2ph - 2p^2h^2 - 1}\sin px\,dp \tag{1.8}$$

Obviously, every solution of this equation is continuous in the interval $[0, a]$, since both the kernel and the free term of the equation are continuous (we recall that by assumption, $f''(r)$ is a continuous function in $[0, R]$ and $a \leqslant R$).

In order to clarify the role of the undetermined constant B in the integral equation (1.7), we express the stresses $\sigma_z(r)$ on the contact area in terms of the function $\chi(x)$. To do this, we replace in the second formula (1.4) $p\alpha(p)$ from the relation (1.6). We obtain

$$\sigma_z(r) = -\int_r^a \frac{\chi(x)\,dx}{\sqrt{x^2-r^2}} - \frac{B}{\sqrt{a^2-r^2}} \tag{1.9}$$

By virtue of the generalized mean value theorem for the continuous function $\chi(x)$ in the interval $[0, a]$ the first term in (1.9) has the limit zero as $r \to a$. For $r \to a$ and $B \neq 0$ the second term increases indefinitely. Consequently, the case $B > 0$ corresponds to the complete indentation of the punch into the layer. But if the punch is not completely indented into the layer, then one has to assume $B = 0$.

From relation (1.6) we obtain the important formula

$$\frac{Q}{2\pi} = \int_0^a x\chi(x)\,dx + Ba \tag{1.10}$$

From (1.10), in the case of the completely indented punch ($a = R$, $B \neq 0$) one can determine the constant B in terms of the force Q which acts on the layer. For the punch which is not completely indented into the layer ($B = 0$), the last formula can be used to determine the radius of the contact area in terms of the known force Q, or vice versa.

We note that for $h \to \infty$ the kernel $K(x, t) \to 0$ uniformly with respect to x and t. Therefore, from Eq. (1.7) we obtain, after a limiting process, the exact expression of the function $\chi(x)$ corresponding to the semispace. Substituting it into the expression which gives the contact stresses $\sigma_z(r)$, we obtain the known solution of the axisymmetric contact problem for the semispace [4].

2. Let $z = f_*(r)$, $r \in [0,R]$ be the equation of the surface of the base of some non-plane punch, which for the value $Q = Q_0$ of the normal force is not completely indented into the layer. Let a_0 be the radius of the contact area between the base of the punch and the layer, corresponding to the force Q_0. We consider the punch with the surface of the base $z = kf_*(r)$, where $0 < k < 1$, and $r \in [0, R]$ which is indented in the layer such that the radius of the contact area is $a = a_0$. Obviously, such an indentation of the punch is possible and it will be incomplete ($B = 0$). It follows from the relations (1.7) and (1.10) that in the latter case $Q = kQ_0$. Therefore, if we increase the force Q up to Q_0, the radius a of the contact area can only increase. Hence it follows that for a given value $Q = Q_0$ of the force, the maximal value of the radius a corresponds to the punch with a plane base, which is not completely indented into the layer (it will be shown below that such an indentation is possible). We denote this maximal value by A. It is easy to show that the quantity A does not depend on the force Q.

Indeed, for a given thickness h of the layer, this quantity has to be taken in such a way that the integral equation (1.7) for the punch with a plane base ($f(r)=0$), not completely indented into the layer ($B = 0$), should admit a nontrivial solution. This means that the kernel (1.8) of the integral equation for $x, t \in [0, A]$ must have in its spectrum the eigenvalue $\lambda = 1$. Consequently, the quantity A depends only on the properties of the kernel (1.8) and does not depend on the force Q, since the kernel does not depend on this force.

From what has been said above about the quantity A it is clear that for any $a < A$ the contact problem for an arbitrary convex punch must have a solution and therefore the integral equation (1.7) corresponding to this punch must also have a solution for $a < A$. Hence it follows that the number A is the smallest of all those values $A > 0$ for which $\lambda = 1$ is an eigenvalue of the kernel $K(x, t)$ $(x, t \in [0, A])$.

The quantity A has been determined by the numerical method of replacing the integral in the homogeneous integral equation (1.7) by a finite sum. In this connection, Gauss' quadrature formulas with three and five nodes have been used. In the process of the computations it has become clear that the kernel $K(x, t)$ $(x, t \in [0, A])$ is non-negative and $u_*(x)$-bounded [3]

$$u_0(x) = \int_0^A K(x, t)\,dt \tag{2.1}$$

For the desired quantity one has obtained the value $A = 1.1\,h$. The quantity A has turned out to be finite, therefore the punch with a plane base, whose cylindrical part has a radius R exceeding A, is incompletely indented into the layer. In other words, for $R > A$ the radius of the contact area of such a punch with the layer is equal to A.

It is interesting to note that the authors of [5], investigating the problem of the unbonded contact between a layer and a semispace under the action of a normal concentrated force, have obtained by another method for the case of an absolutely rigid semispace ($R = \infty$) the value $A = 1.16h$ for the radius of the contact area between the layer and the base, which is close to our value.

The distribution of the stresses on the contact area between the plane punch and the layer, in the case of an incomplete and a complete indentation are given below in the form of the values of $-2\pi a^2 Q^{-1}\sigma_z(r)$

r/a	0	0.05	0.23	0.50	0.77	0.95	1
$a = A$	6.25	6.19	5.25	2.94	1.18	0.35	0
$a = h$	5.28	5.25	4.57	2.78	1.29	0.67	∞

Here the relative values of the stresses $\sigma_z(z)$ are given for $r = 0$ and $r = a\xi_i$ ($i = 1, 2, \ldots, 5$), where ξ_i are the nodes of the Gauss quadrature formula for the interval $[0, 1]$.

Since both the kernel $K(x, t)$ and its adjoint kernel $L(x, t) = K(t, x)$ are nonnegative for $x, t \in [0, a]$, $a < A$ and $u_0(x)$-bounded (the function $u_0(x)$ for the kernel $L(x, t)$ is constructed according to formula (2.1)), one can draw conclusions regarding the character of the dependence between the force Q and its corresponding radius a of the contact area between the layer and an arbitrary punch with a non-plane base. For the sake of simplicity, we will consider that for any value of the force Q the punch identity incompletely into the layer, although the arguments are valid also in the general case.

A nonnegative $u_0(x)$-bounded kernel has [3] a unique positive eigenvalue, which is simple and is the largest in modulus among all the eigenvalues of the kernel. Its corresponding eigenfunction is nonnegative (to within a scalar multiple).

Let $\lambda(a)$ be the positive eigenvalue of the kernel $K(x, t)$ $(x, t \in [0, a])$. Since the real eigenvalues of the kernels $K(x, t)$ and $L(x, t)$ coincide, it follows that $\lambda(a)$ is the positive eigenvalue of the kernel $L(x, t)$ too. The eigenfunctions of the kernels $K(x, t)$ and $L(x, t)$, corresponding to the eigenvalue $\lambda(a)$, will be denoted by $\varphi(x, a)$ and $\psi(x, a)$ respectively. Taking into account that the functions $\varphi(x, a)$ and $\psi(x, a)$ have constant signs (nonnegative) in the interval $[0, a]$, one can prove that the resolvent $\Gamma(x, t, \lambda)$ of the kernel $K(x, t)$ $(x, t \in [0, a])$ has the following structure [3]:

$$\Gamma(x, t, \lambda) = \frac{C\varphi(x, a)\psi(x, a)}{\lambda - \lambda(a)} + \gamma(x, t, \lambda)$$

Here, $\gamma(x, t, \lambda)$ is a regular function in the neighborhood of $\lambda = \lambda(a)$, and C is a positive constant.

We also note that the estimate from below [3] for the positive eigenvalue of the nonnegative kernel $K(x, t)$ $(x, t \in [0, a], a < A)$ implies that $\lambda(a) < \lambda(A) = 1$.

In the integral equation (1.7) we have $\lambda = 1$, therefore its solution can be represented in the form

$$\chi(x) = \Phi(x) + \int_0^a \Phi(t)\Gamma(x, t, 1)\,dt =$$

$$= \frac{C\varphi(x, a)}{1 - \lambda(a)} \int_0^a \Phi(t)\psi(t, a)\,dt + \Phi(x) + \int_0^a \Phi(t)\gamma(x, t, 1)\,dt$$

Here $\Phi(x)$ is the free term of the equation. For a non-plane convex punch we have $\Phi(x) \geqslant 0$.

From the last relation we obtain

$$\int_0 x\chi(x)\,dx = \frac{C}{1 - \lambda(a)} \int_0^a x\varphi(x, a)\,dx \int_0^a \Phi(t)\psi(t, a)\,dt + \ldots$$

For $a \to A$ we have $\lambda(a) \to \lambda(A) = 1$, while $\varphi(x, a)$, $\psi(x, a)$ and $\Phi(x)$ remain nonnegative in $[0, a]$. Therefore

$$\int_0^a x\chi(x)\,dx \to \infty$$

But then, from the relation (1.10) for $a \to A$ we have $Q \to \infty$.

Thus, under the assumptions of the given problem, a non-plane punch cannot be indented into the layer with a finite force in such a way that the radius of the contact area is equal to A.

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